Mathematical Modeling of How We Hear: a Literature Review

Joo Won Park

***Abstract*—This paper is a technical literature review on mathematical modeling of how we hear, especially of the inner ear. This paper covers the relevant history as well as introductory biology of the inner ear. The mathematical model of cochlea introduced in this paper is proposed by M.B. Lesser and D.A. Berkley in 'Fluid Mechanics of of the Cochlea. Part 1' (1972) [4]. Simplified model of the cochlea and the basilar membrane is used to derive a system of partial differential equation with boundary conditions. This paper is a written version of a 75-minutes presentation given in Columbia University's Applied Mathematics Senior Seminar class taught by Professor Christopher Wiggins.**

Keywords—Cochlea, inner ear, basilar membrane, mathematical modeling, partial differential equations, boundary conditions, fluid mechanics.

I. Introduction

The objective of this paper is to review history of the research on the inner ear and introduce one of the mathematical models of the cochlea in order to better understand how the inner ear structure helps us perceive sound. Relevant historical and biological facts about the inner ear and the cochlea are presented, followed by a mathematical modeling of the cochlea proposed by Lesser and Berkley [4].

An integral characteristic of the inner ear investigated in this paper is the frequency selectivity of the basilar membrane in the cochlea. The first portion of the paper covers the discovery of this feature by Georg Von Bekesy [1], and the mathematical modeling in the later portion by Lesser and Berkley [4] is consistent to Bekesy's experimental results.

A. History

In 1862, Hermann Helmholtz published *'The Theory of Sound Sensations'*. This book was the first foundational work on acoustics and perception of sound, which discussed sensation of sound in general such as vibrations, sympathetic resonances, and other phenomena.[6] In this book, Helmholtz identified the basilar membrane as an important part of the hearing process.

After Helmholtz's discovery, scientists' attention geared towards understanding the mechanism of basilar membrane. Anatomically, the nerve ending cells were discovered attached to the basilar membrane. However, it was very difficult to further the research, because the threshold hearing amplitude of basilar membrane vibration was very small, even smaller than the diameter of hydrogen atom [1]. Moreover, experiments on cadaver made it harder to draw accurate measurements as the ear mechanics changed soon after death [6].

The breakthrough was made by a Hungarian biophysicist named Georg Von Bekesy. Bekesy propsed methods of measurement of great precision, by developing a cochlea model with rubber membrane and metal frame. He observed traveling waves along basilar membrane when it was stimulated by sound, and he discovered that basilar membrane served as a frequency selector. Bekesy received the Nobel Prize in Physiology in 1961 for his research on the cochlea. [1]

Bekesy's work inspired theoretical approach to the inner ear mechanics, including mathematical modeling of the inner ear. The goal of the modelng has been being able to demonstrate motion of the basilar membrane that would coincide with experimental results by Bekesy. The model varies by the types of equations and assumptions for the basilar membrane and the fluid in the cochlea (perilymph).

B. Biology

Prior to introducing the mathematical modeling of the cochlea, it is important to understand the relevant anatomy of the ear. The ear system can be divided into three: the external ear, the middle ear, and the inner ear. The external ear focuses sound energy to the eardrum and amplifies the sound pressure. The middle ear has three small interconnected bones, which connect the eardrum to the inner ear.

The oval window, which is at the basal end of the cochlea, is in contact with the stapes, the last of the three bones in the middle ear. Consequentially, the inward movement of the oval window displaces the perilymph (fluid in the cochlea), deforming the basilar membrane that partitions the cochlea. The goal of the mathematical modeling of the cochlea is is to model the manner in which the basilar membrane vibrates in response to sound.

I consulted Dale's Neuroscience textbook, chapter twelve, to write this section. [3]

II. Mathematical Modeling

There are a handful of mathematical models of the cochlea, including models developed by Boer [2] and Viergever[6]. Among these different mathematical models, one developed by Lesser and Berkley [4] is introduced in this paper. This model was chosen because it produced a clear presentation that is consistent to Bksy's results. In addition, dimensionality assumed in Lesser and Berkley's model is suitable. In their model, they assumed a 2-D flow of perilymph (fluid) in the

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cochlea. 1-D flow models have an inherent paradox because fluid ”in contact” with the basilar membrane has a normal velocity to the flow’s direction. 3-D flow models can have the lateral component dropped without qualitatively changing the modeling of the basilar membrane.

The goal of this paper is to derive a system of partial differential equations with boundary conditions of the fluid in a simplified cochlea chamber. The following sections describes the assumptions for the model and essential equations from the fluid dynamics.

A. Assumptions

There are several important assumptions to be made in Lesser and Berkley’s model about the cochlea:

1) The basilar membrane’s motion is primarily controlled by the fluid’s potential. We will be setting up mathematical equations for this variable.
2) Fluid (perilymph) has constant density, irrotational flow, and is inviscid.
3) Regarding cochlea’s structure:
    a) Each point of Basilar Membrane acts as damped harmonic oscillator.
    b) Stapes (bone resting on the oval window) motion determines position of the oval window.
4) The cochlea is the simplified two-chamber model partitioned by the basilar membrane, as seen in Fig. 1: (Note that $0 \leq x \leq L$ and $0 \leq y \leq l$.)

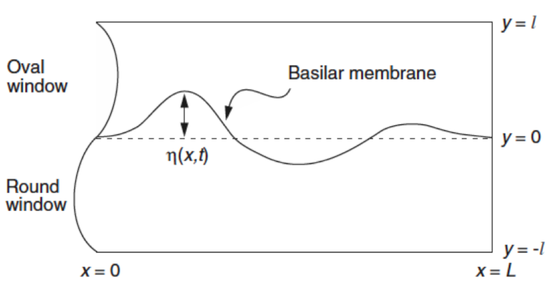

Fig. 1. Simplified Cochlea Model by Lesser and Berkley [4]

The assumptions on the cochlea are integral in setting up equations for the behavior of the basilar membrane. Additionally, The following is the list of variables used in the modeling.(Supscript 1, 2 specifies upper, lower chamber):

1) $\vec{u}$: velocity of fluid (2D)
2) η: y-displacement of BM (the Basilar Membrane)
3) ρ : fluid density (constant)
4) p: fluid pressure
5) Φ: fluid potential ($\nabla\Phi = \vec{u}$)

B. Fluid Equations

There are two equations from the fluid motions that serve as a starting point of the derivation to the PDE system.[5]

1) Equation of Continuity (Conservation of Mass)
For incompressble fluid (fluid density is constant), the conservation of mass leads to the following equation:

$$\frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \vec{u})$$

Under the assumption that the fluid density is constant, the equation becomes:

$$\nabla \cdot (\vec{u}) = 0$$

2) Equation of Motion (Newton’s Second Law)
For inviscid fluid, the Navier-Stokes equation leads to the following equation, assuming that body force (such as gravity) is neglectable:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \nabla)\vec{u} = -\frac{1}{\rho}\nabla p$$

Assuming 2 dimensional linear flow of the fluid, the non-linear term may be ignored, which leads to the following equation:

$$\rho\frac{\partial \vec{u}}{\partial t} + \nabla p = 0$$

3) Defining Flow Potential
Lastly, flow potential can be defined under the assumption that the fluid is irrotational, which mathematically can be represented as $\nabla \times \vec{u} = 0$. Mathematically, we know that curl of gradient is zero. That is, we can define a scalar field such that its gradient is equal to $\vec{u}$. That is, flow potential can be defined:

$$\nabla\Phi = \vec{u}$$

Concept of fluid potential leads to an updated equations of fluid:

Equation of continuity:

$$\Delta\Phi = 0$$

Equation of motion:

$$\rho\frac{\partial \Phi}{\partial t} + p = 0$$

C. Equations for the Structures of the Cochlea

Equations for the fluid in the cochlea (perilymph) is described and simplified. In the following sections, equations regarding the structures of the cochlea is constructed: balance of force in the basilar membrane and equations of the fluid velocity at the boundaries of the cochlea.

1) Equation of the Basilar Membrane: In this section an equation of force (per unit area) for the basilar membrane is constructed. Each point of the basilar membrane behaves like a simple, damped harmonic oscillator. Balance of force (per unit area) leads to the following equation:

$$m(x)\eta_{tt} + r(x)\eta_t + \kappa(x)\eta = p_2 - p_1$$

Note that the characteristic of basilar membrane varies by length. (i.e. Basilar membrane is most stiff at the base and most flexible at the apex.)

$m(x)$, $r(x)$, and $\kappa(x)$ are properties of the basilar membrane depending on the location x. They stand for:

$m(x)$: mass per area

$r(x)$: resistance

$\kappa(x)$: stiffness

2) Boundary Conditions: Now, combining equations from the fluid mechanics and the basilar membrane, a system of partial differential equations with boundary conditions can be derived. Deriving for the upper chamber is sufficient because of the symmetry in the upper and lower chamber of the simplified cochlea model. Note that the functions in the lower chamber are odd in y to the functions in the upper chamber*: (i.e. $\Phi_2 = -\Phi(-y,t)$, $p_2 = -p_1(-y,t)$)

The following is the boundary conditions for the upper chamber ($y \in [0,l], x \in [0,L]$). As demonstrated previously, gradient of flow potential Φ is the flow velocity. That is, $\frac{\partial \Phi}{\partial x}$ and $\frac{\partial \Phi}{\partial y}$ refers to the flow's vertical and horizontal velocity, respectively.

1) Vertical Velocity at $y = 0$

$$\frac{\partial \Phi}{\partial y} = \frac{\partial \eta}{\partial t}$$

The fluid near $y = 0$, or the fluid in contact with the basilar membrane, must behave like the membrane's motion. $\frac{\partial \eta}{\partial t}$ represents the membrane's velocity (y direction).

2) Vertical Velocity at $y = l$

$$\frac{\partial \Phi}{\partial y} = 0$$

At the top of the chamber (at $y = l$), the fluid's vertical velocity is assumed to be 0.

3) Horizontal Velocity at $x = 0$

$$\frac{\partial \Phi}{\partial x} = \frac{\partial F(y,t)}{\partial t}$$

As mentioned in the assumptions regarding cochlea's structure, stapes motion determines position of the oval window (at $x = 0$). F is horizontal **displacement** of oval window, a membrane-covered opening. Thus, $\frac{\partial F(y,t)}{\partial t}$, the horizontal velocity of oval window at point y, determines the fluid's horizontal velocity at the beginning of the chamber. (F is not to be confused as *force*.)

4) Horizontal Velocity at $x = L$

$$\frac{\partial \Phi}{\partial x} = 0$$

At the end of the chamber (at $x = L$), the fluid's horizontal velocity is assumed to be 0.

D. Derivation of PDE and BC system for Flow Potential

In this section, the following sets of equations that has been described by far will be transformed into a more simple system, using technique explained by Lesser and Berkley. First, the frequency-dependent functions F, Φ, p, and η will be expressed in analytic representation. Then, equations 2), 3), and 4) will be combined into one equation in Φ that results in a laplace equation with four boundary conditions.

Fluid Equations

1) $\Delta \Phi = 0$

2) $\rho \dfrac{\partial \Phi}{\partial t} + p = 0$

Basilar Membrane balance of force

3) $m(x)\eta_{tt} + r(x)\eta_t + \kappa(x)\eta = p_2 - p_1 = -2p_1 = -2p$

(Note that p_2 is odd to p_1 in y. The subscript is dropped for simplicity in the last step.)

Boundary Conditions

4) $\dfrac{\partial \Phi}{\partial y} = \dfrac{\partial \eta}{\partial t}$ at $(y = 0)$

5) $\dfrac{\partial \Phi}{\partial y} = 0$ at $(y = l)$

6) $\dfrac{\partial \Phi}{\partial x} = \dfrac{\partial F(y,t)}{\partial t}$ at $(x = 0)$

7) $\dfrac{\partial \Phi}{\partial x} = 0$ at $(x = L)$

E. Analytic Representation

Lesser and Berkley examines steady-state response of the cochlea to the pure tone. F, the displacement of the oval window, is caused by input of single frequency from incoming sound. So, we can express $F(y,t)$ as $F = \hat{F}(y)e^{iwt}$ where w is frequency. Similarly, other frequency-dependent functions Φ, p, and η can be expressed as:

$$\Phi = \hat{\Phi}e^{iwt}, p = \hat{p}e^{iwt}, \eta = \eta_0 e^{iwt}$$

(More details about this can be found in a concept called 'phasor' [7].)

Equations 1) to 7) can be transformed in frequency dependent equations:

Fluid Equations

1) $\Delta \hat{\Phi} = 0$

2) $iw\rho\hat{\Phi} + \hat{p} = 0$

Basilar Membrane balance of force

3) $(-mw^2 + irw + \kappa)\hat{\eta} = -2\hat{p}$

By defining $Z = miw + r + \frac{\kappa}{iw}$, this equation can be expressed in:

$iwZ\hat{\eta} = -2\hat{p}$

Boundary Conditions

4) $\frac{\partial \hat{\Phi}}{\partial y} = iw\eta_0$ at $(y = 0)$

Combining 2) and 3), this equation can be expressed in:

$\dfrac{\partial \hat{\Phi}}{\partial y} = \dfrac{2iw\rho\hat{\Phi}}{Z}$ at $(y = 0)$

5) $\dfrac{\partial\hat{\Phi}}{\partial y} = 0$ at $(y = l)$

6) $\dfrac{\partial\hat{\Phi}}{\partial x} = iw\hat{F} = U_0$ at $(x = 0)$

7) $\dfrac{\partial\hat{\Phi}}{\partial x} = 0$ at $(x = L)$

F. System of Laplace Equation with Neumann Boundary Condition

By far, a system of Laplace Equation with Neumann Boundary Condition has been derived. Scaling x and y by L, Z by $iw\rho L$, and $\hat{\Phi}$ by $U_0 L$ and dropping the hats results in the following simplified system:

1)
$$\Delta\Phi = 0$$

2) at $y = 0$
$$\frac{\partial\Phi}{\partial y} = \frac{2\Phi}{Z}$$

3) at $(y = \frac{l}{L} = \sigma)$
$$\frac{\partial\Phi}{\partial y} = 0$$

4) at $(x = 0)$
$$\frac{\partial\Phi}{\partial x} = 1$$

5) at $(x = 1)$
$$\frac{\partial\Phi}{\partial x} = 0$$

III. Conclusion

As mentioned before, Lesser and Berkley's mathematical model of the cochlea produce results that coincides Bksy's discovery.

Fig. 2. from Lesser and Berkley's paper shows the mathematical results of the displacement at the basilar memrane where each frequency tone peaked in comparison to Bksy's experimental results. (Given membrane's characteristic variables, m, r and κ)

Frequency in Hz	Distance of place from stapes: Békésy (mm)	Distance of place from stapes: Two-dimensional theory (Z from figure 7) (mm)
100	31	33
200	28	28
400	24	23
800	20	17

Fig. 2. Agreement with Bekesy's data [4]

Fig. 3. from Lesser and Berkley's paper graphs Fig. 2. According to the graph, the basilar membrane displays frequency selectivity by its displacement from the stapes. The higher frequency tones' amplitude are peaked closer to the basilar membrane's base, and the lower frequency tones' amplitude are peaked towards the membrane's apex.

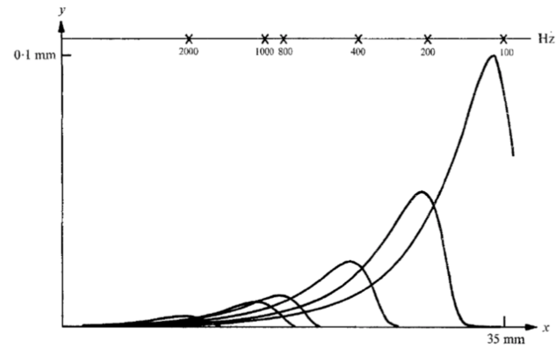

Fig. 3. Unit velocity excitation at stapes, showing wave envelopes [4]

The literature review of the cochlea research gave me a valuable insight of the procedures of mathematical modeling regarding the appropriate assumptions, derivations using substitution and complex transformation, and nondimensionalization by scaling. I hope to continue mathematical modeling research that is pertinent to acoustical sciences in the future.

Appendix A
Solution Suggested by Lesser and Berkley

As an additional information, the solution suggested by Lesser and Berkley is introduced and briefly explained in this section.[4]

The following solution of the fluid flow potential satisfies the equations 1), 3) 5) from section *F*:

$$\Phi = x\left(1 - \frac{1}{2}\right) - \sigma y\left(1 - \frac{y}{2\sigma}\right) + \sum_{n=0}^{\infty} A_n \cosh[n\pi(\sigma - y)]\cos(n\pi x)$$

The complete solution can be found using the second equation 2) by determining the coefficient A_n, using Fast Fourier Transformation algorithm by Cooley and Tukey.[4]

$$A_m \alpha_m = f_m$$

$$\alpha_m = \frac{1}{Z}\cosh(m\pi\sigma) - \frac{1}{2}n\pi\sinh(m\pi\sigma)$$

$$f_m = \sigma\delta_{m0} - \int_0^1 \frac{x(2-x)\cos(m\pi x)}{Z}dx = -\frac{2}{m^2\pi^2}$$

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Joo Won Park was born and raised in South Korea. She came to the United States in August 2013 to pursue her undergraduate studies. During college, Joo Won has worked as a teaching assistant at the department of mathematics in Columbia since her sophomore year. In her free time, she volunteers as an ukulele instructor at a nearby elementary school.